TORSION OF CIRCULAR RODS HAVING LONGITUDINAL NOTCHES OR TEETH AND A CENTRAL CAVITY

(KRUCHENIE KRUGLYKH STERZHNEI, IMEIUSHCHIKH PRODOL'NYE VYTOCHKI ILI ZUBTSY I TSENTRAL'NO RASPOLOZHENNUIU POLOST')

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B.L. ABRAMIAN and A.A. BABLOIAN
(Erevan)

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The problem considered is that of torsion of a circular prismatic rod with symmetrically-located grooves or teeth in the shape of an annular sector, and with a central circular cavity (Fig. 1).

The exact solution of this problem is obtained by a method proposed in [1] and [2].

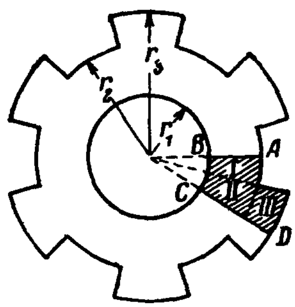

FIG. 1.

The determination of the integration constants leads to the solution of infinite systems of linear equations.

It is shown that the systems obtained are completely regular, that they have an upper limit and that the free terms reduce to zero (for $k \to \infty$).

Stiffnesses and stresses are calculated for two special cases: for six teeth and for one notch.

Approximate solutions for similar complicated rods in torsion have been obtained by Stanesku and Dumitrescu [3] and by Manea [4].

1. For the solution of the given problem a stress function $U(r, \phi)$ is sought which satisfies the equation

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \varphi^2} = -2 \tag{1.1}$$

in the rod section, which reduces to zero on the outside contour, and which takes on a constant value U_0 on the inside contour.

Because of symmetry it is necessary to consider only that part which subtends an angle π/n, where n is the number of notches or teeth (Fig. 2). In order that the solution obtained for the $2n$th part shall be valid for the whole section, it is required that the normal derivative of the function $U(r, \phi)$ be zero on the lines AB and CD. For determination of the function U it is convenient to use the variable

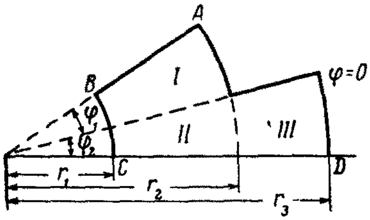

FIG. 2.

$$t = \ln\frac{r}{r_2} \tag{1.2}$$

We seek the function $U(r, \phi)$ in the form

$$U[r(t), \varphi] = \Phi(t, \varphi) - \frac{1}{2} r_2^2 e^{2t} \tag{1.3}$$

where the function $\Phi(t, \phi)$ satisfies the equation

$$\frac{\partial^2 \Phi}{\partial t^2} + \frac{\partial^2 \Phi}{\partial \varphi^2} = 0 \tag{1.4}$$

We assume that the function $\Phi(t, \phi)$ in the part of the region under consideration (Fig. 2) takes on the values

$$\Phi(t, \phi) = \begin{array}{ll} \Phi_1(t, \phi) & \text{in region I} \\ \Phi_2(t, \phi) & \text{in region II} \\ \Phi_3(t, \phi) & \text{in region III} \end{array}$$

By making use of expressions (1.3), (1.5) and the boundary conditions on U, we obtain the following conditions

$$\begin{gathered} \Phi_1(0, \varphi) - \frac{r_2^2}{2} = \Phi_1(-t_1, \varphi) - \Phi_0 = \left(\frac{\partial \Phi_1}{\partial \varphi}\right)_{\varphi=\varphi_1} = 0 \\ \Phi_2(-t_1, \varphi) - \Phi_0 = \left(\frac{\partial \Phi_2}{\partial \varphi}\right)_{\varphi=-\varphi_2} = 0 \\ \Phi_3(t_2, \varphi) - \frac{r_3^2}{2} = \Phi_3(t, 0) - \frac{r_2^2 e^{2t}}{2} = \left(\frac{\partial \Phi_3}{\partial \varphi}\right)_{\varphi=-\varphi_2} = 0 \end{gathered} \tag{1.6}$$

$$\Phi_1(t,\ 0) = \Phi_2(t,\ 0), \qquad \left(\frac{\partial\Phi_1}{\partial\varphi}\right)_{\varphi=0} = \left(\frac{\partial\Phi_2}{\partial\varphi}\right)_{\varphi=0}$$
$$\Phi_2(0,\ \varphi) = \Phi_3(0,\ \varphi), \qquad \left(\frac{\partial\Phi_2}{\partial t}\right)_{t=0} = \left(\frac{\partial\Phi_3}{\partial t}\right)_{t=0} \tag{1.7}$$

for $\Phi_i(t,\ \phi)$ with $i = 1,\ 2,\ 3$, where a is a constant characterizing the groove or tooth width, and where

$$t_1 = \ln\frac{r_2}{r_1}, \quad t_2 = \ln\frac{r_3}{r_2}, \quad \Phi_0 = U_0 + \frac{r_1^2}{2}, \quad \varphi_1 = \frac{\pi}{\alpha n}, \quad \varphi_2 = \frac{\pi}{n} - \frac{\pi}{n\alpha} \tag{1.8}$$

2. Upon solving equation (1.4) by the method of separation of variables, we obtain for the functions $\Phi_i(t,\ \phi)$ the expressions

$$\Phi_1(t,\ \varphi) = \sum_{k=1}^{\infty}\left(A_k^{(1)}\sinh\alpha_k t + B_k^{(1)}\cosh\alpha_k t\right)\sin\alpha_k\varphi + \tag{2.1}$$
$$+ \sum_{k=1}^{\infty}\left(C_k^{(1)}\sinh\beta_k\varphi + D_k^{(1)}\cosh\beta_k\varphi\right)\sin\beta_k t \qquad (0<\varphi<\varphi_1,\ -t_1<t<0)$$
$$\Phi_2(t,\ \varphi) = \sum_{k=1}^{\infty}\left(A_k^{(2)}\sinh\lambda_k t + B_k^{(2)}\cosh\lambda_k t\right)\sin\lambda_k\varphi + \tag{2.2}$$
$$+ \sum_{k=1}^{\infty}\left(C_k^{(2)}\sinh\beta_k\varphi + D_k^{(2)}\cosh\beta_k\varphi\right)\sin\beta_k t \qquad (-\varphi_2<\varphi<0,\ -t_1<t<0)$$
$$\Phi_3(t,\ \varphi) = \sum_{k=1}^{\infty}\left(A_k^{(3)}\sinh\lambda_k t + B_k^{(3)}\cosh\lambda_k t\right)\sin\lambda_k\varphi + \tag{2.3}$$
$$+ \sum_{k=1}^{\infty}\left(C_k^{(3)}\sinh\gamma_k\varphi + D_k^{(3)}\cosh\gamma_k\varphi\right)\sin\gamma_k t \qquad (-\varphi_2<\varphi<0,\ 0<t<t_2)$$

Here

$$\alpha_k = \frac{(2k-1)\pi}{2\varphi_1}, \qquad \beta_k = \frac{k\pi}{t_1}, \qquad \lambda_k = \frac{(2k-1)\pi}{2\varphi_2}, \qquad \gamma_k = \frac{k\pi}{t_2} \tag{2.4}$$

By satisfying the conditions (1.6) and (1.7) we obtain for the integration constants

$$A_k^{(1)} = \frac{r_2^2}{\alpha_k\varphi_1}\coth\alpha_k t_1 - \frac{2\Phi_0}{\alpha_k\varphi_1}\frac{1}{\operatorname{sh}\alpha_k t_1}, \qquad D_k^{(1)} = D_k^{(2)}$$
$$A_k^{(2)} = B_k^{(2)}\coth\lambda_k t_1 - \frac{2\Phi_0}{\lambda_k\varphi_2}\frac{1}{\operatorname{sh}\lambda_k t_1}, \qquad C_k^{(2)} = D_k^{(2)}\tanh\beta_k\varphi_2$$
$$A_k^{(3)} = -B_k^{(2)}\coth\lambda_k t_2 - \frac{r_3^2}{\lambda_k\varphi_2}\frac{1}{\operatorname{sh}\lambda_k t_2}, \qquad B_k^{(3)} = B_k^{(2)} \tag{2.5}$$
$$B_k^{(1)} = \frac{r_2^2}{\alpha_k\varphi_1}, \qquad D_k^{(3)} = \frac{r_2^2\gamma_k}{t_2(4+\gamma_k^2)}\left[1 + (-1)^{k+1}e^{2t_2}\right]$$

$$C_k^{(1)} = - D_k^{(2)} \tanh \beta_k \varphi_1, \qquad C_k^{(3)} = \frac{r_2^2 \gamma_k}{t_2 (4+\gamma_k^2)} [1 + (-1)^{k+1} e^{2t_2}] \tanh \gamma_k \varphi_2$$

and for determination of the constants $B_k^{(2)}$ and $D_k^{(2)}$ we obtain a combination of two infinite systems of linear equations.

Upon applying the change in variables

$$\lambda_k \varphi_2 B_k^{(2)} = Y_k m - r_2^2, \qquad \beta_k t_1 D_k^{(2)} = X_k - r_2^2 - 2\Phi_0 (-1)^{k+1} \tag{2.6}$$

where m is a constant number to be determined later, the combination is obtained in the form

$$X_k = \sum_{p=1}^{\infty} a_{kp} Y_p, \qquad Y_k = \sum_{p=1}^{\infty} b_{kp} X_p + Q_k \qquad (k = 1, 2, \ldots, \infty) \tag{2.7}$$

where

$$a_{kp} = \frac{2m\beta_k}{\varphi_2 (\tanh \beta_k \varphi_1 + \tanh \beta_k \varphi_2)(\lambda_p^2 + \beta_k^2)}$$
$$b_{kp} = \frac{2\lambda_k}{m t_1 (\coth \lambda_k t_1 + \coth \lambda_k t_2)(\beta_p^2 + \lambda_k^2)} \tag{2.8}$$

$$Q_k = - \frac{2\lambda_k r_2^2}{m(\lambda_k^2 - 4)(\coth \lambda_k t_1 + \coth \lambda_k t_2)} \left[1 + \frac{\lambda_k^2 - 4}{2\lambda_k^2 t_1} \left(\frac{2\Phi_0}{r_2^2} - 1 \right) + \right.$$
$$\left. + \frac{2}{\lambda_k} \left(\coth \lambda_k t_2 - \frac{e^{2t_2}}{\sinh \lambda_k t_2} \right) \right] \tag{2.9}$$

3. We show that the systems (2.7) are completely regular [5]. The following inequalities hold for sums of the moduli of the coefficients in the systems (2.7):

$$\sum_{p=1}^{\infty} |a_{kp}| = \frac{2m\beta_k}{\varphi_2 (\tanh \beta_k \varphi_1 + \tanh \beta_k \varphi_2)} \sum_{p=1}^{\infty} \frac{1}{\lambda_p^2 + \beta_k^2} \leqslant m \tag{3.1}$$

$$\sum_{p=1}^{\infty} |b_{kp}| = \frac{2\lambda_k}{m t_1 (\coth \lambda_k t_1 + \coth \lambda_k t_2)} \sum_{p=1}^{\infty} \frac{1}{\beta_p^2 + \lambda_k^2} \leqslant \frac{1}{2m} \tag{3.2}$$

For this, the values

$$\sum_{p=1}^{\infty} \frac{1}{\beta_p^2 + \lambda_k^2} = \frac{t_1}{2\lambda_k} \left(\coth \lambda_k t_1 - \frac{1}{\lambda_k t_1} \right) \leqslant \frac{t_1}{2\lambda_k}, \qquad \sum_{p=1}^{\infty} \frac{1}{\lambda_p^2 + \beta_k^2} = \frac{\varphi_2}{2\beta_k} \tanh \beta_k \varphi_2$$

have been used. The constant number m is selected from the inequalities

$$m=\frac{1}{2m}, \quad \text{or} \quad m=\frac{\sqrt{2}}{2} \tag{3.3}$$

Then, in accordance with (3.1), we shall have

$$\sum_{p=1}^{\infty} |a_{kp}| \leqslant \frac{\sqrt{2}}{2}, \qquad \sum_{p=1}^{\infty} |b_{kp}| \leqslant \frac{\sqrt{2}}{2}$$

Thus, the systems (2.7) are shown to be completely regular.

It is easy to see from (2.9) that the free terms in the systems (2.7) have an upper limit and that they approach zero for $k \to \infty$.

4. By substitution of the values of the coefficients from (2.5) into (2.1)-(2.3) and after replacing the coefficients $B_k^{(2)}$ and $D_k^{(2)}$ by X_k and Y_k, in accordance with (2.6), we obtain after certain transformations the following expressions for the functions $\Phi_i(t, \phi)$

$$\Phi_1(t, \varphi)=\frac{r_2^2}{2}+\frac{r_2^2-2\Phi_0}{2t_1}t+\frac{1}{t_1}\sum_{k=1}^{\infty}\frac{X_k\cosh\beta_k(\varphi_1-\varphi)}{\beta_k\cosh\beta_k\varphi_1}\sin\beta_k t \tag{4.1}$$
$$(0\leqslant\varphi\leqslant\varphi_1, \quad -t_1\leqslant t\leqslant 0)$$

$$\Phi_2(t, \varphi)=\frac{r_2^2}{2}+\frac{r_2^2-2\Phi_0}{2t_1}t+\frac{1}{t_1}\sum_{k=1}^{\infty}\frac{X_k\cosh\beta_k(\varphi_2+\varphi)}{\beta_k\cosh\beta_k\varphi_2}\sin\beta_k t+$$
$$+\frac{m}{\varphi_2}\sum_{k=1}^{\infty}\frac{Y_k\sinh\lambda_k(t_1+t)}{\lambda_k\sinh\lambda_k t_1}\sin\lambda_k\varphi \qquad \begin{pmatrix}-\varphi_2\leqslant\varphi\leqslant 0\\ -t_1\leqslant t\leqslant 0\end{pmatrix} \tag{4.2}$$

$$\Phi_3(t, \varphi)=\frac{r_2^2}{2}+\frac{r_3^2-r_2^2}{2t_2}t+\frac{m}{\varphi_2}\sum_{k=1}^{\infty}\frac{Y_k\sinh\lambda_k(t_2-t)}{\lambda_k\sinh\lambda_k t_2}\sin\lambda_k\varphi- \tag{4.3}$$
$$-\frac{4r_2^2}{t_2}\sum_{k=1}^{\infty}\frac{1+(-1)^{k+1}e^{2t_2}}{\gamma_k(\gamma_k^2+4)}\frac{\cosh\gamma_k(\varphi_2+\varphi)}{\cosh\gamma_k\varphi_2}\sin\gamma_k t \qquad \begin{pmatrix}-\varphi_2\leqslant\varphi\leqslant 0\\ 0\leqslant t\leqslant t_2\end{pmatrix}$$

5. For determination of the constant Φ_0, the theorem of Bredt for the circulation of shear stress in torsion

$$\int_{\Gamma_0} T_s ds = 2G\tau\Omega_0 \tag{5.1}$$

is employed, where Γ_0 is the inside contour of the section, $\Omega_0 = \pi r_1^2$ is the area bounded by Γ_0, G is the shear modulus, τ the angle of twist per unit length, T_s the projection of shear stress at any point of the contour Γ_0 in the tangential direction of the contour

$$T_s=\tau_{z\varphi}=-\frac{G\tau e^{-t}}{r_2}\left(\frac{\partial\Phi}{\partial t}-r_2^2e^{2t}\right) \quad \text{for } t=-t_1 \tag{5.2}$$

Substituting (5.2) into (5.1) gives

$$\int_{-\varphi_2}^{\varphi_1}\left(\frac{\partial\Phi}{\partial t}\right)_{t=-t_1}d\varphi=\int_{-\varphi_2}^{0}\left(\frac{\partial\Phi_2}{\partial t}\right)_{t=-t_1}d\varphi+\int_{0}^{\varphi_1}\left(\frac{\partial\Phi_1}{\partial t}\right)_{t=-t_1}d\varphi=0 \tag{5.3}$$

By substitution of the values of the functions Φ_1 and Φ_2 from (4.1) and (4.2) into these relations and then performing the integrations, we obtain the following formula:

$$(\varphi_1 + \varphi_2)(r_2^2 - 2\Phi_0) - \frac{2m}{\varphi_2} \sum_{p=1}^{\infty} \frac{Y_p}{\lambda_p^2} = 0 \tag{5.4}$$

after a certain transformation to determine the constant Φ_0.

The unknown coefficients Y_p appearing in (5.4) are determined from the completely regular infinite systems of linear equations (2.7) and are expressed in terms of the constant Φ_0.

Substitution of the values of the unknown Y_p's obtained from (2.7) into (5.4) and solving for Φ_0 yields its value.

6. The torsional stiffness of a doubly-connected profile is determined from the formula

$$C = 2G\left[-U_0^*\Omega_0^* + U_0\Omega_0 + \iint_{\Omega} U d\Omega\right] = 2G\left[-U_0^*\Omega_0^* + U_0\Omega_0 + 2n\iint_{\Omega^*} U d\Omega\right] \tag{6.1}$$

where U_0^* is the value of the stress function on the outside contour, U_0 the value on the inside contour; Ω_0^* and Ω_0 are the areas bounded by the outside and inside contours, respectively; Ω is the region of the rod section; and Ω^* is that part of the region included in the angle π/n.

By substitution of (1.3) into (6.1) and by using expressions (1.5) and (4.1)-(4.3), we obtain after integration the following formula:

$$\begin{aligned} C = 2G\Big\{&\frac{\pi}{4}(r_2^4 - r_1^4) + \frac{\pi}{4t_1}(2\Phi_0 - r_2^2)(r_2^2 - r_1^2) + \\ &+ \frac{n\varphi_2}{4}(r_3^4 - r_2^4) - \frac{n\varphi_2}{4t_2}(r_3^2 - r_2^2)^2 + \\ &+ 2nr_2^2\Big[\frac{m}{\varphi_2}\sum_{k=1}^{\infty}\frac{Y_k}{\lambda_k(4-\lambda_k^2)}\left(\coth\lambda_k t_1 + \coth\lambda_k t_2 - \frac{e^{-2t_1}}{\sinh\lambda_k t_1} - \frac{e^{2t_2}}{\sinh\lambda_k t_2}\right) - \\ &- \frac{4r_2^2}{t_2}\sum_{k=1}^{\infty}\frac{\tanh\gamma_k\varphi_2}{\gamma_k(4+\gamma_k^2)^2}[1 + (-1)^{k+1}e^{2t_2}]^2 - \\ &- \frac{1}{t_1}\sum_{k=1}^{\infty}\frac{X_k[1 + (-1)^{k+1}e^{-2t_1}]}{\beta_k(4+\beta_k^2)}(\tanh\beta_k\varphi_1 + \tanh\beta_k\varphi_2)\Big]\Big\} \end{aligned} \tag{6.2}$$

Upon passing to the limit $t_1 \to \infty$ $(r_1 = 0)$, we obtain from (6.2) a formula for the stiffness of a solid circular section with teeth:

$$C = 2G\left\{\frac{\pi r_2^4}{4} + \frac{n\varphi_2}{4}(r_3^4 - r_2^4) - \frac{n\varphi_2}{4t_2}(r_3^2 - r_2^2)^2 + \right.$$

$$+ 2nr_2{}^2\Big[\frac{m}{\varphi_2}\sum_{k=1}^{\infty}\frac{Y_k}{\lambda_k(4-\lambda_k{}^2)}\Big(1+\coth\lambda_k t_2-\frac{e^{2t_2}}{\sinh\lambda_k t_2}\Big)- \tag{6.3}$$

$$-\frac{4r_2{}^2}{t_2}\sum_{k=1}^{\infty}\frac{\tanh\gamma_k\varphi_2}{\gamma_k(4+\gamma_k{}^2)^2}[1+(-1)^{k+1}e^{2t_2}]^2-\frac{1}{\pi}\int_0^{\infty}\frac{X(z)(\tanh z\varphi_1+\tanh z\varphi_2)}{z(4+z^2)}\,dz\Big]\Big\}$$

Here the unknown coefficients $X(z)$ and Y_k must be determined from the systems obtained from (2.7) by passing to the limit $t_1 \to \infty$. By passing to the limit $t_2 \to 0$ $(r_3 = r_2)$, we get a formula for the stiffness of a circular rod with a central circular cavity:

$$C = \frac{G\pi}{2}(r_2{}^4 - r_1{}^4) \tag{6.4}$$

For this case we employ the values

$$X_k = Y_k = 0, \qquad \Phi_0 = \frac{r_2{}^2}{2} \tag{6.5}$$

which are obtained from (2.7) and (5.4) by passing to the corresponding limit $t_2 = 0$.

7. For numerical examples we consider the cases where the rod has six external teeth together with a circular cavity (Fig. 1) with the values

$$\frac{r_2}{r_1} = 1.964, \qquad \frac{r_3}{r_2} = 1.2523, \qquad \varphi_1 = \varphi_2 = \frac{1}{12}\pi \tag{7.1}$$

and also where the rod has one external notch together with a central cavity (Fig. 3) and with the values

$$\frac{r_2}{r_1} = 1.964, \qquad \frac{r_3}{r_2} = 1.2523, \qquad \varphi_1 = 0.125 = 7°09'43''$$

$$\varphi_2 = 3.0166 = 172°50'17'' \tag{7.2}$$

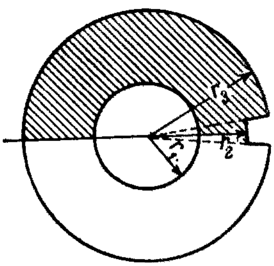

FIG. 3.

Upon solving the infinite systems (2.7) for these cases we obtain too large and too small values for the unknown coefficients X_k and Y_k.

From Formulas (5.4) and (6.2) we obtain also the values of the constant Φ_0 and the stiffness C. These values are presented in Table 1.

In this same table are also included for comparison the stiffnesses of a rod with sections in the shape of circular rings having ratios $b/a = r_2 r_1 = 1.964$ and $b/a = r_3 r_1 = 2.4595$; the first ring is obtained from the toothed profile (Fig. 1) by removal of the teeth $(r_3 = r_2)$, and the second ring is obtained from the notched profile (Fig. 3) by filling up the notch $(r_2 = r_3)$.

TABLE 1.

	$\frac{r_2}{r_1}=1{,}964$	Six teeth (Fig. 1) too small	Six teeth (Fig. 1) too large	Six teeth (Fig. 1) mean	One notch (Fig. 3) too small	One notch (Fig. 3) too large	One notch (Fig. 3) mean	$\frac{r_2}{r_1}=2.4595$
$2\Phi_0/r_2^2$	1.0	1.237	1.238	1.2375	1.5232	1.5252	1.5242	1.5683
C/Gr_2^4	1.4652	2.929	2.941	2.935	3.6208	3.6281	3.6245	3.7579

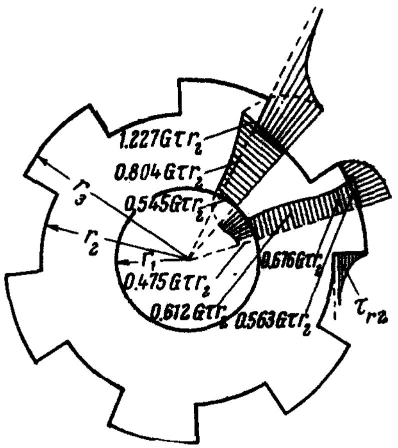

FIG. 4.

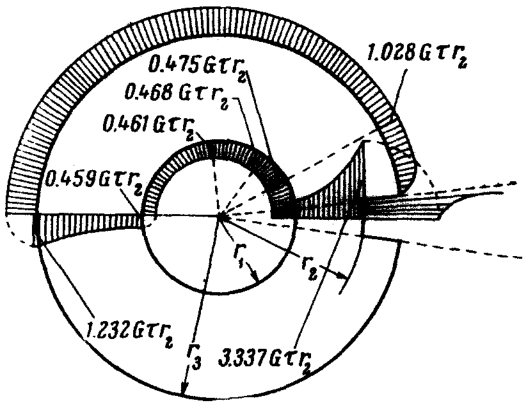

FIG. 5.

Stresses for these cases are calculated with the aid of expressions (4.1)-(4.3) according to the formulas

$$\tau_{zr}(t,\varphi)=\frac{e^{-t}}{r_2}\frac{\partial\Phi}{\partial\varphi}G\tau,\qquad \tau_{z\varphi}(t,\varphi)=-\frac{e^{-t}}{r_2}\left(\frac{\partial\Phi}{\partial t}-r_2^2e^{2t}\right)G\tau. \tag{7.3}$$

TABLE 2.

$\frac{\tau_{z\varphi}}{G\tau r_2}$	$(0,\varphi_1)$	$\left(-\frac{t_1}{2},\varphi_1\right)$	$(-t_1,\varphi_1)$	$(t_2,-\varphi_2)$	$\left(\frac{t_2}{2},-\varphi_2\right)$	$\left(-\frac{t_1}{2},-\varphi_2\right)$	$(-t_1,-\varphi_2)$
Too small	1.2271	0.8036	0.5446	0.6755	0.5632	0.6114	0.4741
Too large	1.2278	0.8043	0.5459	0.6757	0.5636	0.6123	0.4754
Mean	1.2274	0.8040	0.5452	0.6756	0.5634	0.6118	0.4748

TABLE 3.

$\frac{\tau_{z\varphi}}{G\tau r_2}$	$(0, \varphi_1)$	$\left(-t_1, -\frac{\varphi_2}{8}\right)$	$\left(-t_1, -\frac{\varphi_2}{4}\right)$	$\left(-t_1, -\frac{\varphi_2}{2}\right)$	$(-t_1, -\varphi_2)$	$(t_2, -\varphi_2)$	$\left(t_2, -\frac{\varphi_2}{8}\right)$
Too small	3.3207	0.4705	0.4641	0.4586	0.4559	1.2265	1.0241
Too large	3.3526	0.4791	0.4726	0.4627	0.4612	1.2379	1.0309
Mean	3.3366	0.4748	0.4683	0.4606	0.4586	1.2322	1.0275

Values of the stress $\tau_{z\phi}$ at different points (t, ϕ) are presented in Tables 2 and 3 for the cases with six teeth and with one notch. The stress distributions are shown in Figs. 4 and 5.

8. The torsion of a circular prismatic rod having a central circular cavity with symmetrical longitudinal grooves or teeth (Fig. 6) is considered. The exact solution of this problem may be obtained in the same manner as in Sections 1-6.

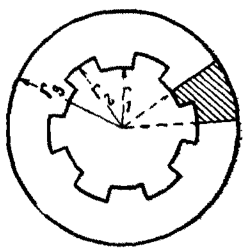

FIG. 6.

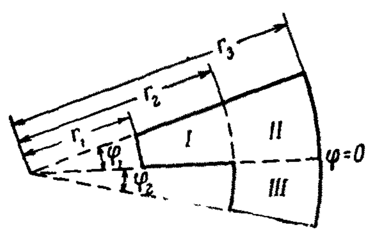

FIG. 7.

For the solution we assume that the stress function $U(r, \phi)$ on the inside contour of the section reduces to zero, and that on the outside contour it takes on a constant value $U_0{}^*$.

We seek a solution in the form of (1.3) on the assumption that the function $\Phi(t, \phi)$ takes on the values (1.5); the regions I, II, and III are shown in Fig. 7.

The functions $\Phi_i(t, \phi)$ $(i = 1, 2, 3)$ satisfy Equation (1.4) in the respective regions and also the following boundary conditions and continuity conditions:

$$\Phi_1(-t_1, \varphi) - \frac{r_1{}^2}{2} = \Phi_1(t, 0) - \frac{r_2{}^2 e^{2t}}{2} = \left(\frac{\partial \Phi_1}{\partial \varphi}\right)_{\varphi=\varphi_1} = 0$$

$$\Phi_2(t_2, \varphi) - \Phi_0{}^* = \left(\frac{\partial \Phi_2}{\partial \varphi}\right)_{\varphi=\varphi_1} = 0$$

$$\Phi_3(t_2,\varphi)-\Phi_0^*=\Phi_3(0,\varphi)-\frac{r_2^2}{2}=\left(\frac{\partial\Phi_3}{\partial\varphi}\right)_{\varphi=-\varphi_2}=0 \tag{8.1}$$
$$\Phi_1(0,\varphi)=\Phi_2(0,\varphi),\qquad \left(\frac{\partial\Phi_1}{\partial t}\right)_{t=0}=\left(\frac{\partial\Phi_2}{\partial t}\right)_{t=0}$$
$$\Phi_2(t,0)=\Phi_3(t,0),\qquad \left(\frac{\partial\Phi_2}{\partial\varphi}\right)_{\varphi=0}=\left(\frac{\partial\Phi_3}{\partial\varphi}\right)_{\varphi=0}$$

where

$$t_1=\ln\frac{r_2}{r_1},\qquad t_2=\ln\frac{r_3}{r_2},\qquad \Phi_0^*=U_0^*+\frac{r_3^2}{2} \tag{8.2}$$

9. Upon solving Equation (1.4) for the functions $\Phi_i(t,\phi)$ $(i=1,2,3)$ by the method of separation of variables and upon satisfying conditions (8.1), we obtain expressions for these functions

$$\Phi_1(t,\varphi)=\frac{r_2^2}{2}+\frac{r_2^2-r_1^2}{2t_1}t+\frac{m}{\varphi_1}\sum_{k=1}^{\infty}\frac{Y_k\sinh\alpha_k(t_1+t)}{\alpha_k\sinh\alpha_k t_1}\sin\alpha_k\varphi+$$
$$+\frac{4r_2^2}{t_1}\sum_{k=1}^{\infty}\frac{1+(-1)^{k+1}e^{-2t_1}}{\beta_k(\beta_k^2+4)}\frac{\cosh\beta_k(\varphi_1-\varphi)}{\cosh\beta_k\varphi_1}\sin\beta_k t\quad(0\leqslant\varphi\leqslant\varphi_1,\ -t_1\leqslant t\leqslant 0) \tag{9.1}$$

$$\Phi_2(t,\varphi)=\frac{r_2^2}{2}+\frac{2\Phi_0^*-r_2^2}{2t_2}t+\frac{m}{\varphi_1}\sum_{k=1}^{\infty}\frac{Y_k\sinh\alpha_k(t_2-t)}{\alpha_k\sinh\alpha_k t_2}\sin\alpha_k\varphi+$$
$$+\frac{1}{t_2}\sum_{k=1}^{\infty}\frac{X_k\cosh\gamma_k(\varphi_1-\varphi)}{\gamma_k\cosh\gamma_k\varphi_1}\sin\gamma_k t\quad(0\leqslant\varphi\leqslant\varphi_1,\quad 0\leqslant t\leqslant t_2) \tag{9.2}$$

$$\Phi_3(t,\varphi)=\frac{r_2^2}{2}+\frac{2\Phi_0^*-r_2^2}{2t_2}t+\frac{1}{t_2}\sum_{k=1}^{\infty}\frac{X_k\cosh\gamma_k(\varphi_2+\varphi)}{\gamma_k\cosh\gamma_k\varphi_2}\sin\gamma_k t$$
$$(-\varphi_2\leqslant\varphi\leqslant 0,\quad 0\leqslant t\leqslant t_2) \tag{9.3}$$

where

$$\alpha_k=\frac{(2k-1)\pi}{2\varphi_1},\qquad \beta_k=\frac{k\pi}{t_1},\qquad \gamma_k=\frac{k\pi}{t_2},\qquad m=\frac{\sqrt{2}}{2} \tag{9.4}$$

The integration constants X_k and Y_k entering into (9.1)-(9.3) are determined from the following combination of infinite systems of linear equations:

$$X_k=\sum_{p=1}^{\infty}a_{kp}Y_p,\qquad Y_k=\sum_{p=1}^{\infty}b_{kp}X_p+Q_k\qquad(k=1,2,\ldots) \tag{9.5}$$

where

$$a_{kp}=\frac{2m\gamma_k}{\varphi_1(\tanh\gamma_k\varphi_1+\tanh\gamma_k\varphi_2)(\gamma_k^2+\alpha_p^2)} \tag{9.6}$$
$$b_{kp}=\frac{2\alpha_k}{mt_2(\coth\alpha_k t_1+\coth\alpha_k t_2)(\alpha_k^2+\gamma_p^2)}$$

$$Q_k = -\frac{2\alpha_k r_2^2}{m\,(\alpha_k^2 - 4)\,(\coth \alpha_k t_1 + \coth \alpha_k t_2)} \times$$
$$\times \left[1 + \frac{\alpha_k^2 - 4}{2\alpha_k^2 t_2}\left(1 - \frac{2\Phi_0^*}{r_2^2}\right) - \frac{2}{\alpha_k}\left(\coth \alpha_k t_1 - \frac{e^{-2t_1}}{\sinh \alpha_k t_1}\right)\right] \tag{9.7}$$

Systems (9.5) are completely regular, since the following inequalities hold:

$$\sum_{p=1}^{\infty} |a_{kp}| = \frac{2m\gamma_k}{\varphi_1(\tanh\gamma_k\varphi_1 + \tanh\gamma_k\varphi_2)} \sum_{p=1}^{\infty} \frac{1}{\alpha_p^2 + \gamma_k^2} \leqslant m = \frac{\sqrt{2}}{2} \tag{9.8}$$
$$\sum_{p=1}^{\infty} |b_{kp}| = \frac{2\alpha_k}{m t_2\,(\coth \alpha_k t_1 + \coth \alpha_k t_2)} \sum_{p=1}^{\infty} \frac{1}{\gamma_p^2 + \alpha_k^2} \leqslant \frac{1}{2m} = \frac{\sqrt{2}}{2}$$

The constant quantity Φ_0^* entering into expressions (9.1)-(9.3) is determined from the Bredt's theorem concerning the circulation of shear stress in torsion.

This theorem, as applied to the external contour of the section, is expressed by the relation

$$\int_{-\varphi_1}^{0} \left(\frac{\partial \Phi_3}{\partial t}\right)_{t=t_2} d\varphi + \int_{0}^{\varphi_1} \left(\frac{\partial \Phi_2}{\partial t}\right)_{t=t_2} d\varphi = 0 \tag{9.9}$$

and, after carrying out the integration, takes on the form

$$(\varphi_1 + \varphi_2)(2\Phi_0^* - r_2^2) - \frac{2m}{\varphi_1} \sum_{p=1}^{\infty} \frac{Y_p}{\alpha_p^2} = 0 \tag{9.10}$$

The unknown coefficients Y_p, determined from the infinite systems of (9.5), are expressed in terms of Φ_0^*, and in order to find Φ_0^* the relation (9.10) must be solved for Φ_0^*.

10. By making use of Formula (6.1), substituting (1.3) therein and performing the integrations, we obtain the following formula for the stiffness of a circular rod having a central cavity with symmetrical longitudinal grooves:

$$C = 2G\left\{\frac{\pi}{4}(r_3^4 - r_2^4) - \frac{\pi}{4t_2}(2\Phi_0^* - r_2^2)(r_3^2 - r_2^2) + \frac{n\varphi_1}{4}(r_2^4 - r_1^4) - \right.$$

$$- \frac{n\varphi_1}{4t_1}(r_2^2 - r_1^2)^2 + 2nr_2^2\left[\frac{m}{\varphi_1}\sum_{k=1}^{\infty} \frac{Y_k}{\alpha_k(\alpha_k^2 - 4)}\left(\coth \alpha_k t_1 + \coth \alpha_k t_2 - \right.\right.$$

$$-\frac{e^{-2t_1}}{\sinh\alpha_k t_1}-\frac{e^{2t_2}}{\sinh\alpha_k t_2}\Bigg)-\frac{r_2^2}{t_1}\sum_{k=1}^{\infty}\frac{\tanh\beta_k\varphi_1}{\beta_k(\beta_k^2+4)^2}\left[1+(-1)^{k+1}e^{-2t_1}\right]^2+$$

$$+\frac{1}{t_2}\sum_{k=1}^{\infty}\frac{X_k\left[1+(-1)^{k+1}e^{2t_2}\right]}{\gamma_k(\gamma_k^2+4)}\left(\tanh\gamma_k\varphi_1+\tanh\gamma_k\varphi_2\right)\Bigg]\Bigg\} \qquad (10.1)$$

As an example, we cite the value of the stiffness in torsion for a circular rod having a central cavity with six symmetrical longitudinal grooves (Fig. 6), when

$$\frac{r_2}{r_1}=1.964, \qquad \frac{r_3}{r_2}=1.964, \qquad \varphi_1=\varphi_2=\frac{1}{12}\pi \qquad (10.2)$$

Values of the constant $\Phi_0{}^*$ and the stiffness C for this case, calculated from Formulas (9.10) and (10.1), are presented in Table 4.

TABLE 4.

	$\frac{r_3}{r_2}=1.964$	With 6 notches or teeth			$\frac{r_3}{r_1}=3.8573$
		too small	too large	mean	
$2\Phi_0^*/r_2^2$	1.0	0.9168	0.9195	0.9182	0.2592
C/Gr_2^4	21.801	22.062	22.408	22.074	23.266

For comparison, the table gives also the stiffness of a rod with a section in the form of circular rings with ratios $b/a = r_3/r_2 = 1.964$ and $b/a = r_3/r_1 = 3.8573$.

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